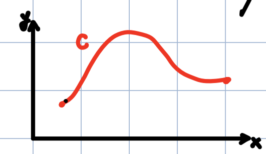
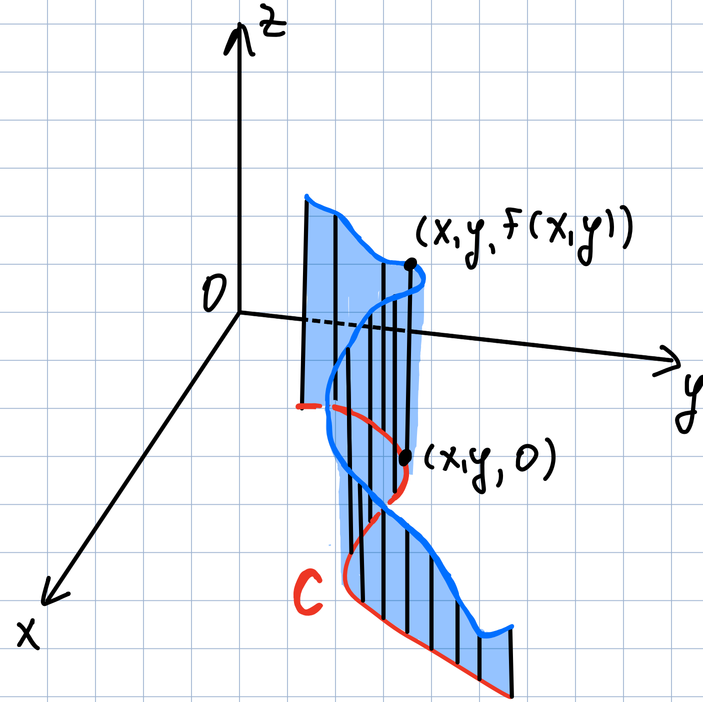
LAST TIME: Line integrals of functions

C-plane curve:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad a \le t \le b$$





Line (curve) integral of $f(x,y)$ along C:

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Rmk: If $f(x,y) \ge 0$ then $\int_C f(x,y)\,ds$ represents the area of one side of the "fense"

Line integrals w.r.t. x, y:

$$\int_C f(x,y)\,d\underline{\underline{x}} = \int_a^b f(x(t), y(t))\, \underline{x'(t)}\,dt$$

$$\int_C f(x,y)\,d\underline{\underline{y}} = \int_a^b f(x(t), y(t))\, \underline{y'(t)}\,dt$$

If C is the form of a (thin) wire with linear density $\rho(x,y)$, then

$$m = \int_C \rho(x,y)\,ds \quad \text{- mass}$$

$$\bar{x} = \frac{1}{m} \int_C x\,\rho(x,y)\,ds$$

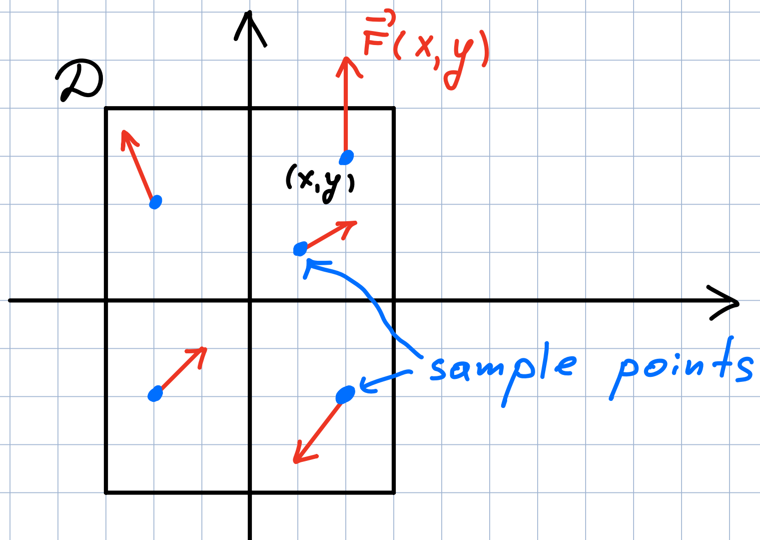
$$\bar{y} = \frac{1}{m} \int_C y\,\rho(x,y)\,ds$$

$(\bar{x}, \bar{y})$ - center of mass.

Vector fields, line integrals

- A vector field on $\mathbb{R}^2$ is a function $\vec{F}(x,y)$ assigning to each point $(x,y) \in D$ a 2D vector $\vec{F}(x,y)$

region in $\mathbb{R}^2$



$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$$

component functions

$$= P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$$

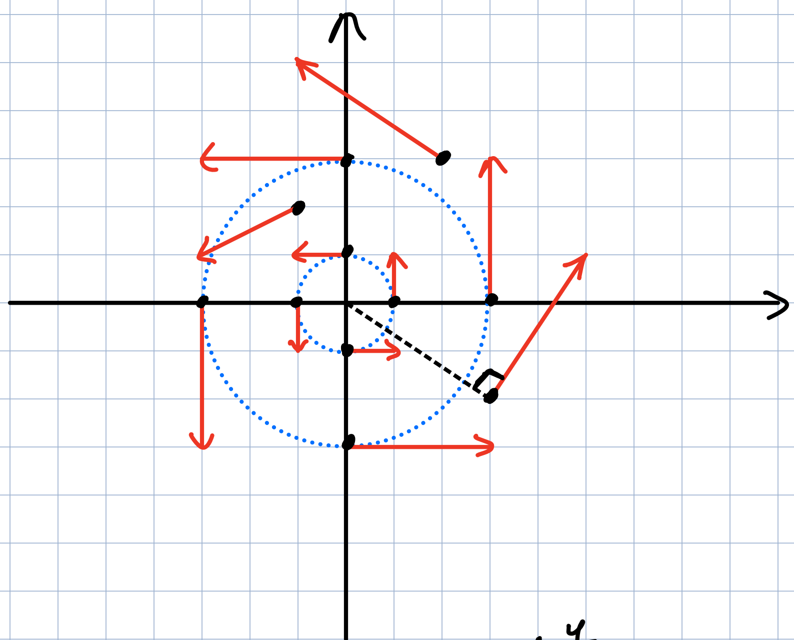
Rmk: $P(x,y)$, $Q(x,y)$ – scalars

- One calls P, Q – scalar fields

- A vector field on $\mathbb{R}^3$ is a function $\vec{F}(x,y,z)$ assigning to each point $(x,y,z) \in E$ a 3D vector $\vec{F}(x,y,z)$

solid in space

Ex: $F(x,y) = -y\vec{i} + x\vec{j}$ - vector field on $\mathbb{R}^2$. Sketch it.

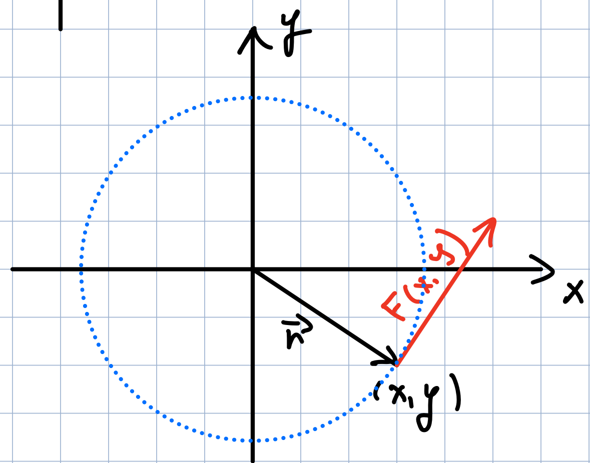


In fact: each arrow tangent to circle centered at origin

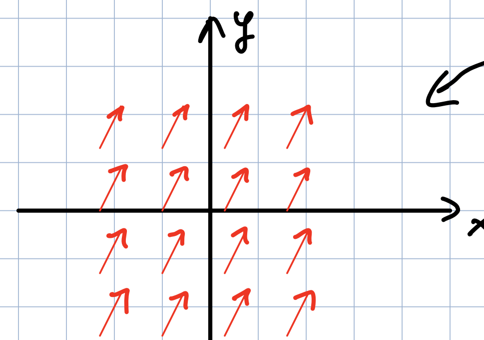
$$\vec{r} \cdot \vec{F}(\vec{r}) = (x\vec{i} + y\vec{j}) \cdot (-y\vec{i} + x\vec{j})$$
$$= -xy + xy = 0$$

$$\Rightarrow \vec{F}(\vec{r}) \perp \vec{r}$$

$\Rightarrow \vec{F}(\vec{r})$ tangental to the circle with center at the origin and radius $r = \sqrt{x^2+y^2}$.



Ex: $F(x,y) = \vec{i} + 2\vec{j}$



One can rescale all vectors in pict. by (same) factor.

Gravitational force field of Earth:

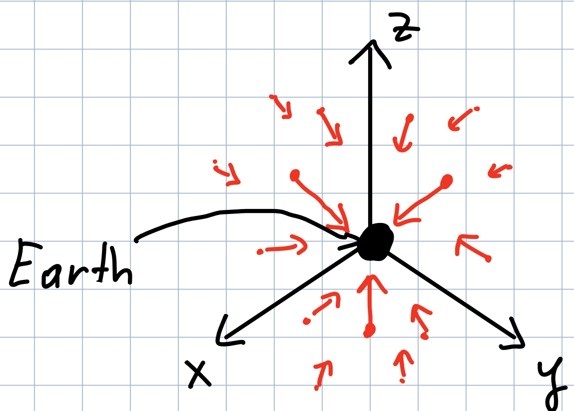
Gravitational force acting on the object at $\vec{r} = \langle x, y, z \rangle$

$$F(\vec{r}) = -\frac{mMG}{|\vec{r}|^3}\vec{r}$$

M - mass of the earth

m - object's mass

$$F(x, y, z) = \frac{-mMGx}{(x^2+y^2+z^2)^{3/2}}\vec{i} + \frac{-mMGy}{(x^2+y^2+z^2)^{3/2}}\vec{j} + \frac{-mMGz}{(x^2+y^2+z^2)^{3/2}}\vec{k}$$

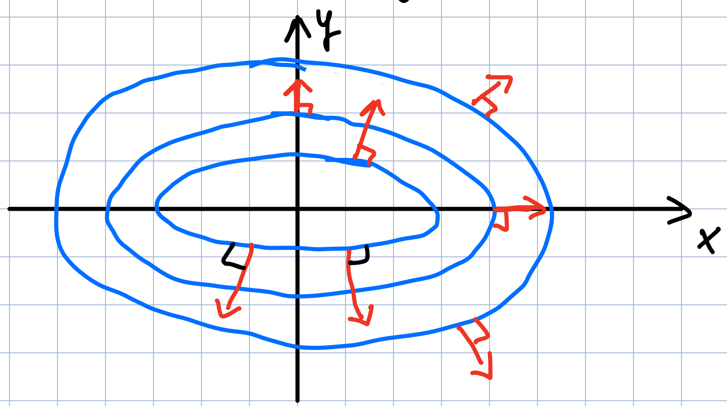


Gradient vector field:

$$\nabla f(x,y) = \langle f_x(x,y),\ f_y(x,y) \rangle$$

$$\nabla f(x,y,z) = \langle f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z) \rangle$$

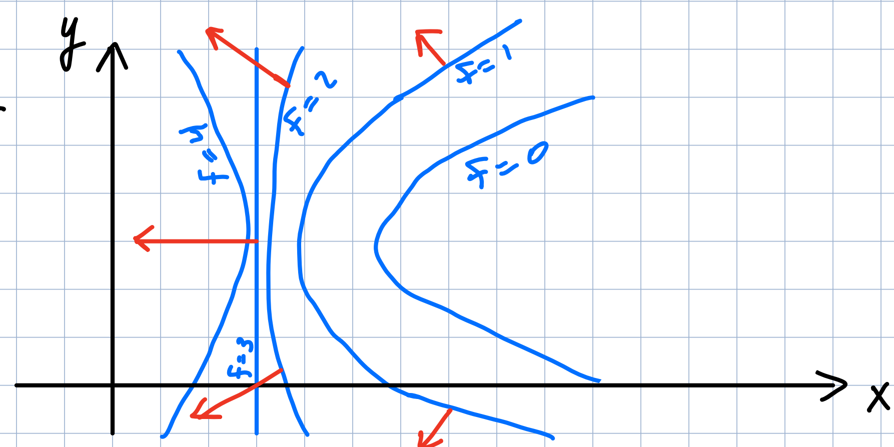
Ex: $f = x^2 + 4y^2$ $\qquad \nabla f = \langle 2x, 8y \rangle$

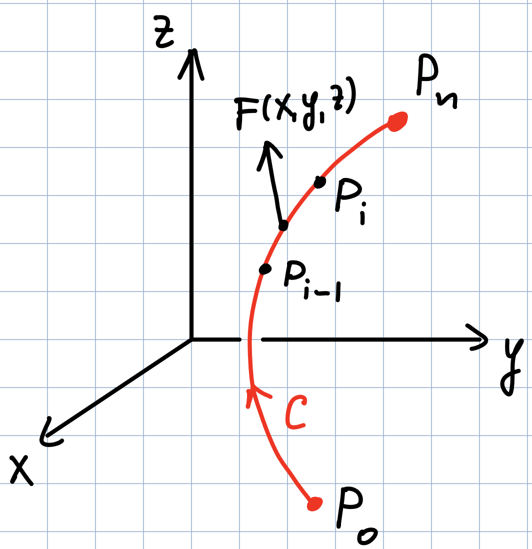


Rmk: Gradient vectors are perpendicular to level lines (curves).

Rmk: Grad. vectors are long where level curves are close to each other.

Generally:



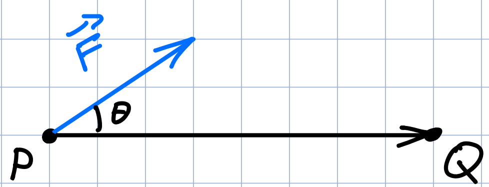


C-curve, $\vec{F}(x,y,z)$ - Force field

Work W done by Force $\vec{F}$ while moving a particle along curve C:

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(x_i^*, y_i^*, z_i^*) \cdot \Delta S_i \underbrace{\vec{T}}_{\text{unit tangent vector to } C}(t_i^*)$$

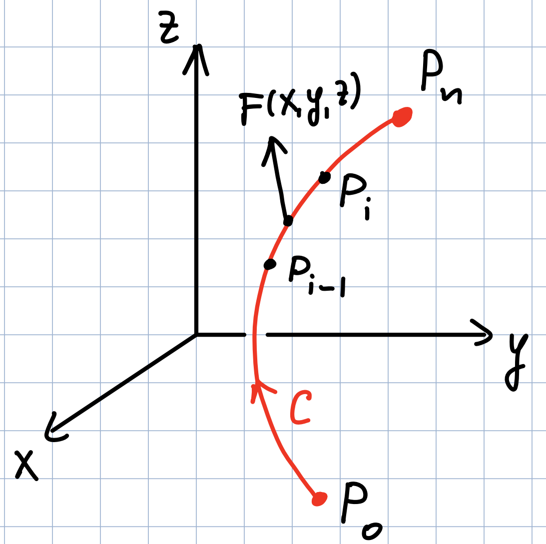
Reminder: The force $\vec{F}$ moves the object from P to Q.



The work done by this force:

$$W = \underbrace{(|F| \cos \theta)}_{\text{component of the force along displacement}} \overbrace{|PQ|}^{\text{distance moved}}$$

$$\Rightarrow \quad W = \vec{F} \cdot \overrightarrow{PQ}$$



C - curve, $\vec{F}(x,y,z)$ - Force field

Work W done by Force $\vec{F}$ while moving a particle along curve C:

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(x_i^*, y_i^*, z_i^*) \cdot \Delta s_i \underbrace{\vec{T}(t_i^*)}_{\text{unit tangent vector to } C}$$

$$= \int_C \vec{F}(x,y,z) \cdot \underbrace{\vec{T}(x,y,z)}_{\frac{\vec{r}'(t)}{|\vec{r}'(t)|}} \underbrace{ds}_{|\vec{r}'(t)|\,dt}$$

$$\Rightarrow \boxed{W = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt}$$

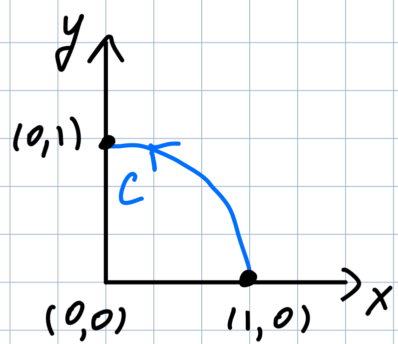
- Line integral of a vector field $\vec{F}$ along a curve C given by $\vec{r}(t)$, $a \le t \le b$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \underbrace{\vec{F}(\vec{r}(t))}_{\vec{F}(x(t),\,y(t),\,z(t))} \cdot \vec{r}'(t)\,dt = \int_C \vec{F} \cdot \vec{T}\,ds$$

Ex: Find the work W done by force field $\vec{F}(x,y) = x^2\vec{i} - xy\vec{j}$ in moving a particle along quarter-circle

$$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j}, \quad 0 \le t \le \frac{\pi}{2}$$

Sol:



$$W = \int_0^{\pi/2} \underbrace{\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)}_{\langle \cos^2 t,\, -\cos t \sin t\rangle \cdot \langle -\sin t,\, \cos t\rangle} \, dt$$

$$= \int_0^{\pi/2} (-\cos^2 t \sin t - \cos^2 t \sin t)\, dt = -2\int_0^{\frac{\pi}{2}} \cos^2 t \sin t \, dt$$

$$= 2\, \frac{\cos^3 t}{3} \Big|_{t=0}^{t=\frac{\pi}{2}} = -\frac{2}{3}$$

In space: $\int_C \vec{F} \cdot d\vec{r} = \int_C P(x,y,z)\,dx + \int_C Q(x,y,z)\,dy + \int_C R(x,y,z)\,dz$

$\langle P, Q, R \rangle$

Ex: Find $W = \int_C \vec{F} \cdot dr$, where $\vec{F}(x,y,z) = \langle xy, yz, zx \rangle$

and C is given by $\begin{array}{l} x = t \\ y = t^2 \\ z = t^3 \end{array}$, $0 \le t \le 1$

Sol: $\vec{r} = \langle t, t^2, t^3 \rangle$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

$\vec{F}(\vec{r}(t)) = \langle t^3, t^5, t^4 \rangle$

So, $w = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^1 (t^3 + 2t^6 + 3t^6)\,dt = \frac{1}{4} + \frac{5}{7} = \frac{27}{28}$

Remarks:

Arc length - Lecture 9

Tangent vector (line) to space curve - Lecture 8

Vector eq. of the curve - Lecture 7

Vector/parametric equation of a line - Lecture 5

Dot product - Lecture 3